Squeeze tomography of quantum states

2004 J. Phys. A: Math. Gen. 37 8529

(http://iopscience.iop.org/0305-4470/37/35/009)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 8529–8544 PII: S0305-4470(04)72924-8

Squeeze tomography of quantum states

Octavio Castaños, Ramón López-Peña, Margarita A Man'ko[1] **and Vladimir I Man'ko**[1]

Instituto de Ciencias Nucleares, UNAM, AP 70-543, 04510 Mexico, DF, Mexico

Received 3 December 2003
Published 17 August 2004
Online at stacks.iop.org/JPhysA/37/8529
doi:10.1088/0305-4470/37/35/009

Abstract
A new type of tomogram, squeeze tomograms, of quantum states of light, is introduced, which is based on measuring photon statistics after the action of a squeeze operator on the states' density matrix. The introduced tomograms are photon number probability functions depending on two real parameters which are a scaling factor and a rotation angle. The rotation angle and scaling factor label the reference frame in the phase space of field quadratures in which the photon statistics is measured. The expressions of the squeeze tomogram in terms of the Wigner function and density matrix in quadrature representation are obtained. The examples of squeezed states, even and odd coherent states and thermal states are considered in detail. The multimode generalization is discussed.

PACS numbers: 03.65.Ca, 03.65.Ta, 03.65.Wj, 42.50.Ar, 42.50.Dv

1. Introduction

The quantum states are described either by wavefunctions [1] (pure states) or by a density matrix [2, 3]. The attempts to find a description of the quantum states which more closely resembles the classical picture gave rise to the quasidistribution functions in phase space of field quadratures such as the Wigner function [4], Sudarshan–Glauber P-function [5, 6] and Husimi Q-function [7]. Recently it was understood that the states can be associated with the standard probability distribution functions. This understanding emerged when the relation between the marginal distribution function for photon homodyne quadrature (optical tomogram) and the Wigner function was found [8, 9].

Optical tomography of quantum states was used to measure the quantum states of squeezed light [10, 11]. The optical tomograms depending on a rotation angle parameter were generalized [12–14] to the case of symplectic tomograms of quantum states, which depend on a field quadrature and two additional real parameters. The symplectic tomograms

[1] On leave from: P N Lebedev Physical Institute, Moscow, Russia.

are probability distribution functions for a field quadrature measured in a scaled and rotated reference frame of quadrature phase space. The symplectic and optical tomograms depend on random continuous variables (homodyne quadrature components). There exists another tomographic scheme to measure the quantum states. This scheme uses probability distributions of discrete random variable $n = 0, 1, 2, \dots,$ which has the physical meaning of number of photons. Photon number tomography was introduced in [15–17]. The electromagnetic field amplitude of the initial quantum state of light in photon number tomography is shifted by an arbitrary amount $\alpha = \alpha_1 + \mathrm{i}\alpha_2$. The photon number distribution function measured after the shifting contains complete information on the initial quantum state of the photon. Another tomographic scheme, which uses a discrete random variable for measuring quantum states, is based on spin tomography [18–21]. In spin tomography the discrete random variable is the spin projection m, $-j < m < j$, i.e., the random variable changes in a finite domain.

In the standard theory of quantum states, there exist different representations, e.g., position representation, momentum representation, energy representation. The pure state is described by a state vector in Hilbert space but the choice of a specific basis in the Hilbert space for representing the state vector in terms of its components with respect to the chosen basis determines the concrete representation. One can choose, for example, the discrete basis of Fock states. Also one can choose the continuous basis of coherent states. All the different basis vectors are connected by some linear transform. The description of mixed quantum states is given in terms of a density operator. For the density operator, one can also use different basis vectors to construct the density matrix. In all cases, the study of different representations (different basis vectors and corresponding linear transforms) is important to clarify different aspects of the quantum states. An analogous situation is seen in the tomographic approach to the description of quantum states. In the description of quantum states the wavefunctions corresponding to different representations are connected by some integral transforms, e.g., by Fourier transform as in the case of position and momentum representations. Although all the representations contain equivalent information, one uses the specific one (like using the position representation) which is most appropriate for a given problem. Due to this, it is useful to find different representations and to study their properties; the same is true in the tomographic approach. The motivation is to study different possibilities to associate with quantum states the probability-distribution functions. Quasidistribution functions such as the Husimi Q-function or the Wigner function are associated with so-called symbols of density operators. The symbols of arbitrary operators are functions which are in one-to-one correspondence with the operators. To reproduce properties of the associative product of operators, the symbols of operators have a nonstandard product. This product (called the star-product) is associative but, in general, noncommutative. Tomograms (or tomographic symbols) are a particular case of star-product quantization.

The aim of our paper is to introduce a new tomographic representation which we called ‘squeeze tomography’ and to study the relation of the squeeze tomograms to symplectic tomograms, the Wigner function and density matrix in the position representation. The squeeze tomogram uses the discrete random variable $n = 0, 1, 2, \dots,$ which is the photon number analogous to the case of photon-number tomography. But instead of using the shift of the photon amplitude, squeeze tomography is based on scaling the photon amplitude.

The paper is organized as follows. In section 2 we review symplectic tomography and optical tomography schemes. In section 3 we review the Wigner function and the density matrix in the position representation. In section 4 we introduce squeeze tomography and study the connections of squeeze tomograms with the Wigner functions, optical and symplectic tomograms and the density matrix in the position representation. The examples of the squeeze tomograms for the coherent states [6], even and odd coherent states [22], thermal states and squeezed states [23, 24] are considered in section 5. The extension of the scheme to the multimode electromagnetic field is discussed in section 6. Perspectives and conclusion are given in section 7.

2. Symplectic and optical tomograms

The state of a quantum system is described by a Hermitian density operator $\hat{\varrho}$ which satisfies

$$\mathrm{Tr}\,\hat{\varrho} = 1, \tag{1}$$

and has the property of nonnegativity

$$\varrho_n \geqslant 0, \tag{2}$$

where ϱ_n are the eigenvalues of the density operator $\hat{\varrho}$. For a pure state, the density operator is a projector, i.e.,

$$\hat{\varrho}^2 = \hat{\varrho}. \tag{3}$$

For continuous variables (position or field quadrature) one can introduce the optical tomographic probability distribution [8–10]

$$\mathcal{W}_{\mathrm{opt}}(X, \theta) = \langle \delta\,(X - \cos\theta\,\hat{q} - \sin\theta\,\hat{p})\rangle. \tag{4}$$

This positive probability distribution, called an optical tomogram, is normalized for a normalized quantum state, i.e.,

$$\int_{-\infty}^{\infty} \mathrm{d}X\, \mathcal{W}_{\mathrm{opt}}(X, \theta) = 1. \tag{5}$$

It is important that the optical tomogram contains the same information about the states as the density operator does. The optical tomogram determines the quantum state completely. It can be considered as particular characteristics of the state analogous to the density matrix in position representation. The optical tomogram can be extended to become the symplectic tomogram

$$\mathcal{W}_{\mathrm{sym}}(X, \mu, \nu) = \langle \delta(X - \mu\hat{q} - \nu\hat{p})\rangle. \tag{6}$$

Here μ and ν are real numbers. The random variable X can be treated as the position (field quadrature) measured in the scaled and rotated reference frame of phase space. The symplectic tomogram also determines the quantum state completely. It can be used as characteristics of the state instead of the density matrix in the position (or other) representation. The Dirac δ-function is defined by its Fourier decomposition as

$$\delta(X - \mu\hat{q} - \nu\hat{p}) = \frac{1}{2\pi} \int \mathrm{d}k\, \mathrm{e}^{\mathrm{i}k(X - \mu\hat{q} - \nu\hat{p})}.$$

The parameters λ and θ, where

$$\mu = \mathrm{e}^{\lambda} \cos\theta, \qquad \nu = \mathrm{e}^{-\lambda} \sin\theta, \tag{7}$$

describe scaling and rotation, respectively. Equation (6) can be rewritten using the expression for the Wigner quasidistribution function of the state $W(q, p)$ [4]

$$\mathcal{W}_{\mathrm{sym}}(X, \mu, \nu) = \int \frac{\mathrm{d}q\,\mathrm{d}p}{2\pi} W(q, p)\, \delta(X - \mu q - \nu p). \tag{8}$$

This expression has the inverse

$$W(q, p) = \int \frac{\mathrm{d}X\,\mathrm{d}\mu\,\mathrm{d}\nu}{2\pi}\, \mathrm{e}^{\mathrm{i}(X - \mu q - \nu p)} \mathcal{W}_{\mathrm{sym}}(X, \mu, \nu). \tag{9}$$

The density operator $\hat{\varrho}$ can be expressed in terms of the symplectic tomogram $\mathcal{W}_{\rm sym}(X,\mu,\nu)$ as [13]

$$\hat{\varrho} = \frac{1}{2\pi}\int \mathrm{d}X\,\mathrm{d}\mu\,\mathrm{d}\nu\,\mathrm{e}^{\mathrm{i}(X-\mu\hat{q}-\nu\hat{p})}\mathcal{W}_{\rm sym}(X,\mu,\nu). \tag{10}$$

The optical tomogram can be related to the Wigner function

$$\mathcal{W}_{\rm opt}(X,\theta) = \int \frac{\mathrm{d}q\,\mathrm{d}p}{2\pi} W(q,p)\,\delta(X-\cos\theta q-\sin\theta p). \tag{11}$$

According to [9], the inversion formula can be written in terms of the Radon transform

$$W(q,p) = \frac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\pi}\mathrm{d}\eta\,\mathrm{d}X\,\mathrm{d}\theta\,|\eta|\,\mathrm{e}^{\mathrm{i}\eta(X-q\cos\theta-p\sin\theta)}\mathcal{W}_{\rm opt}(X,\theta). \tag{12}$$

The integrations must be understood in the sense of generalized functions [24]. Thus, we obtained that both the optical tomogram and the symplectic tomogram can be used for a description of quantum states analogously to how the Wigner function is used.

3. Wigner function properties

The Wigner function is connected with the density matrix in the position representation

$$W(q,p) = \int \mathrm{d}u\,\mathrm{e}^{-\mathrm{i}pu}\varrho\left(q+\frac{u}{2},q-\frac{u}{2}\right), \tag{13}$$

with the inverse relation

$$\varrho(x,x') = \langle x|\hat{\varrho}|x'\rangle = \frac{1}{2\pi}\int \mathrm{d}p\,\mathrm{e}^{\mathrm{i}p(x-x')}W\left(\frac{x+x'}{2},p\right). \tag{14}$$

Usually the Wigner function is considered as the appropriate function to describe the quantum state. But the symplectic tomogram as well as the optical one can also be associated with the quantum state since they contain the equivalent information which is in the Wigner function or density matrix. The Wigner function provides marginal distribution functions

$$P(q) = \int \frac{\mathrm{d}p}{2\pi} W(q,p), \tag{15}$$

$$P(p) = \int \frac{\mathrm{d}q}{2\pi} W(q,p). \tag{16}$$

In equations (15) and (16), $P(q)$ and $P(p)$ are the position and momentum probability densities, respectively. One can rewrite equation (15) in the form

$$\int \frac{\mathrm{d}p}{2\pi} W(q,p) = \varrho(q,q), \tag{17}$$

where $\varrho(q,q)$ is the diagonal element of the density matrix in position representation. In view of equations (15) and (16), the Wigner function is similar to the positive probability distribution on the phase space of a classical system. But the Wigner function can take negative values too. It means that the Wigner function cannot be interpreted as a standard probability distribution on the phase space. Thus if one wants to associate with the quantum state a fair probability distribution, one can use the optical $\mathcal{W}_{\rm opt}(X,\theta)$ and symplectic $\mathcal{W}_{\rm sym}(X,\mu,\nu)$ tomograms, which provide examples of probability densities of the field quadrature component X either in a rotated reference frame (optical tomogram) or in a rotated and scaled reference frame (symplectic case).

4. Squeeze tomograms

In this section we introduce another type of tomogram which we call the squeeze tomogram $\mathcal{W}_{\rm sq}(n, \mu, \nu)$. Here $n = 0, 1, 2, \ldots$ has the physical meaning of the number of photons in the quantum state of light under consideration. We define the tomogram of the state with the density operator $\hat{\varrho}$ by the relation

$$\begin{aligned}\mathcal{W}_{\rm sq}(n, \mu, \nu) &= \langle n|\hat{\boldsymbol{\mathcal{S}}}(\mu, \nu)\,\hat{\varrho}\hat{\boldsymbol{\mathcal{S}}}^{\dagger}(\mu, \nu)|n\rangle \\ &= \langle n|\hat{S}(\lambda)\hat{R}(\theta)\,\hat{\varrho}\hat{R}^{\dagger}(\theta)\hat{S}^{\dagger}(\lambda)|n\rangle. \end{aligned} \tag{18}$$

Here $\hat{\boldsymbol{\mathcal{S}}}(\mu, \nu) = \hat{S}(\lambda)\hat{R}(\theta)$, where $\hat{S}(\lambda)$ and $\hat{R}(\theta)$ are the squeeze and rotation operators, respectively. They have the form

$$\hat{S}(\lambda) = \exp\left[\frac{\mathrm{i}\lambda}{2}(\hat{q}\hat{p} + \hat{p}\hat{q})\right], \tag{19}$$

$$\hat{R}(\theta) = \exp\left[\frac{\mathrm{i}\theta}{2}(\hat{q}^2 + \hat{p}^2)\right]. \tag{20}$$

The scaling parameter λ and rotation angle θ are connected with the symplectic transform parameters μ and ν by equation (7).

The squeeze tomogram can be interpreted as the diagonal matrix element in a Fock basis of the scaled and rotated density operator

$$\hat{\varrho}^{\mu\nu} = \hat{\boldsymbol{\mathcal{S}}}(\mu, \nu)\hat{\varrho}\hat{\boldsymbol{\mathcal{S}}}^{\dagger}(\mu, \nu). \tag{21}$$

Since the squeezing and rotation are unitary operators the Hermitian nonnegative density operator $\hat{\varrho}^{\mu\nu}$ has positive diagonal matrix elements in the Fock basis. These matrix elements (tomograms) have the physical meaning of photon distribution functions in the state described by the density operator $\hat{\varrho}^{\mu\nu}$. To measure the tomogram one has to take the initial photon state with the density operator $\hat{\varrho}$. Then one needs to rotate the quadratures as is done in the homodyne detection scheme. The rotated state has to be squeezed by applying the squeezing operator $\hat{S}^{\dagger}(\lambda)$. Measuring the photon statistics in the obtained state with the density operator $\hat{\varrho}^{\mu\nu}$, one gets the squeeze tomogram $\mathcal{W}_{\rm sq}(n, \mu, \nu)$. This tomogram is the normalized probability distribution of the discrete random variable n. The tomogram is normalized, satisfying the equality

$$\sum_{n=0}^{\infty}\mathcal{W}_{\rm sq}(n, \mu, \nu) = 1. \tag{22}$$

The tomogram depends on the number of photons n and two real parameters μ and ν. The number of parameters is sufficient to characterize the quantum state completely since it is determined by the Wigner function depending on two real variables q and p.

4.1. Squeeze tomogram and density operator

Let us find the connection of the introduced squeeze tomogram with other characteristics of photon quantum states, e.g., with the density operator (density matrix) in the position representation. This connection can be presented in the form of an integral transform of the density matrix

$$\begin{aligned}\mathcal{W}_{\rm sq}(n, \mu, \nu) &\equiv \mathcal{W}_{\rm sq}(n, \lambda, \theta) \\ &= \int \mathrm{d}x\,\mathrm{d}y\,\varrho(x, y)\,\mathcal{K}(x, y, n, \mu, \nu). \end{aligned} \tag{23}$$

The kernel of the integral transform has the form

$$\mathcal{K}(x, y, n, \mu, \nu) = \frac{1}{\sqrt{\pi(\mu^2+\nu^2)2^n n!}} H_n\left(\frac{x}{\sqrt{\mu^2+\nu^2}}\right) H_n\left(\frac{y}{\sqrt{\mu^2+\nu^2}}\right) \times \exp\left\{-\mathrm{i}\frac{x^2}{2}\left[\frac{\sqrt{2}}{1-\sqrt{1-4\mu^2\nu^2}} - \frac{\mu+\mathrm{i}\nu}{\nu(\mu^2+\nu^2)}\right] + \mathrm{i}\frac{y^2}{2}\left[\frac{\sqrt{2}}{1-\sqrt{1-4\mu^2\nu^2}} - \frac{\mu-\mathrm{i}\nu}{\nu(\mu^2+\nu^2)}\right]\right\}, \tag{24}$$

where H_n denotes the Hermite polynomial of order n. The derivation of this formula is given in appendix A.

4.2. Squeeze tomogram and Wigner function

One can find the relation of the squeeze tomogram to the Wigner function. To do this we express the density matrix in terms of the Wigner function using equations (13), (14) (see details in appendix B). The connection of the squeeze tomogram with the Wigner function can be presented in the integral form

$$\mathcal{W}_{\mathrm{sq}}(n, \mu, \nu) = \int \mathrm{d}q\, \mathrm{d}p\, W(q, p)\, \mathcal{K}_W(q, p, n, \mu, \nu). \tag{25}$$

The kernel of the integral transform has the form

$$\mathcal{K}_W(q, p, n, \mu, \nu) = \frac{(-1)^n}{\pi} \exp(-|z|^2/2) L_n(|z|^2), \tag{26}$$

with

$$|z|^2 = \frac{2q^2}{\mu^2+\nu^2} + 2(\mu^2+\nu^2)\left[p - \left(\frac{\sqrt{2}}{1-\sqrt{1-4\mu^2\nu^2}} - \frac{\mu}{\nu(\mu^2+\nu^2)}\right) q\right]^2. \tag{27}$$

For $\nu = 1, \mu = 0$ (or $\theta = \pi/2, \lambda = 0$), which means that there is no squeezing, and a $\pi/2$ rotation, the obtained kernel coincides with the Wigner function of the Fock state (given in [27]).

4.3. Squeeze tomogram and symplectic tomogram

The symplectic tomograms can be written within the framework of the star-product quantization [28]. Then it is associated with the set of operators

$$\hat{U}(\vec{x}) = \delta(X - \hat{\boldsymbol{\mathcal{S}}}^\dagger(\mu, \nu)\, \hat{q}\, \hat{\boldsymbol{\mathcal{S}}}(\mu, \nu)), \tag{28}$$

$$\hat{D}(\vec{x}) = \frac{1}{2\pi} \exp\{\mathrm{i}\,(X - \mu\hat{q} - \nu\hat{p})\}, \tag{29}$$

with $\vec{x} = (X, \mu, \nu)$. The symplectic tomogram is the tomographic symbol of the density operator and it is given by

$$\mathcal{W}(\vec{x}) = f_{\hat{\varrho}}(\vec{x}) = \mathrm{Tr}\{\hat{\varrho}\hat{U}(\vec{x})\},$$

and the density operator,

$$\hat{\varrho} = \int \mathrm{d}X\, \mathrm{d}\mu\, \mathrm{d}\nu \mathcal{W}_{\hat{\varrho}}(X, \mu, \nu) \hat{D}(X, \mu, \nu).$$

As we mentioned in the introduction, the tomographic symbol of the operator is the function, which represents all the properties of the operator. Suppose that there is another mapping described by the vector $\vec{y} = (n, \mu', \nu')$, and sets of operators $\hat{U}'(\vec{y})$ and $\hat{D}'(\vec{y})$ such that the squeeze tomogram is another tomographic symbol of the density operator

$$\mathcal{W}(\vec{y}) = \phi_{\hat{\varrho}}(\vec{y}) = \mathrm{Tr}\{\hat{\varrho}\hat{U}'(\vec{y})\},$$

and the inverse relation

$$\hat{\varrho} = \sum_n \int \mathrm{d}\mu' \,\mathrm{d}\nu' \,\mathcal{W}_{\hat{\varrho}}(n, \mu', \nu')\hat{D}'(n, \mu', \nu').$$

The operator $\hat{U}'(n, \mu', \nu')$ is given by the expression

$$\hat{U}'(n, \mu', \nu') = \delta(n - \hat{\mathcal{S}}^\dagger(\mu', \nu')\hat{a}^\dagger\hat{a}\hat{\mathcal{S}}(\mu', \nu')), \tag{30}$$

where $\hat{a}^\dagger, \hat{a}$ are boson creation and annihilation operators. The expression for $\hat{D}(n, \mu', \nu')$ is under current research.

Two different symbols of the same operator can be related through the expression

$$\phi_A(\vec{y}) = \int \mathrm{d}\vec{x}\, f_A(\vec{x})\,\mathrm{Tr}\{\hat{D}(\vec{x})\hat{U}'(\vec{y})\}.$$

Therefore

$$\mathcal{W}_{\mathrm{sq}}(n, \mu', \nu') = \int \mathrm{d}X\,\mathrm{d}\mu\,\mathrm{d}\nu\,\mathcal{W}_{\mathrm{sym}}(X, \mu, \nu)\mathcal{K}_S(n, \mu', \nu', X, \mu, \nu) \tag{31}$$

where the transformation kernel is given by

$$\mathcal{K}_{\mathcal{S}}(n, \mu', \nu', X, \mu, \nu) = \mathrm{Tr}\left\{\frac{1}{2\pi}\exp[\mathrm{i}(X - \mu\hat{q} - \nu\hat{p})]\,\delta(n - \hat{\mathcal{S}}^\dagger(\mu', \nu')\hat{a}^\dagger\hat{a}\hat{\mathcal{S}}(\mu', \nu'))\right\}. \tag{32}$$

Substituting the expressions

$$\delta(n - \hat{\mathcal{S}}^\dagger(\mu', \nu')\,\hat{a}^\dagger\hat{a}\hat{\mathcal{S}}(\mu', \nu')) = \hat{\mathcal{S}}^\dagger(\mu', \nu')\,\delta(n - \hat{a}^\dagger\hat{a})\,\hat{\mathcal{S}}(\mu', \nu') \tag{33}$$

and

$$\hat{\mathcal{S}}(\mu, \nu)\hat{q}\hat{\mathcal{S}}^\dagger(\mu, \nu) = \mu\hat{q} + \nu\hat{p} \tag{34}$$

into the equation for the transformation kernel, we can establish that

$$\begin{aligned}\mathcal{K}_{\mathcal{S}}&(n, \mu', \nu', X, \mu, \nu)\\ &= \mathrm{Tr}\left\{\frac{1}{2\pi}\hat{\mathcal{S}}(\mu, \nu)\exp(X - \hat{q})\,\hat{\mathcal{S}}^\dagger(\mu, \nu)\,\hat{\mathcal{S}}^\dagger(\mu', \nu')\delta(n - \hat{a}^\dagger\hat{a})\hat{\mathcal{S}}(\mu', \nu')\right\}\\ &= \frac{1}{2\pi}\langle n|\hat{\mathcal{S}}(\tilde{\mu}, \tilde{\nu})\exp(X - \hat{q})\,\hat{\mathcal{S}}^\dagger(\tilde{\mu}, \tilde{\nu})|n\rangle,\end{aligned} \tag{35}$$

where

$$\hat{\mathcal{S}}(\tilde{\mu}, \tilde{\nu}) = \hat{\mathcal{S}}(\mu', \nu')\hat{\mathcal{S}}(\mu, \nu)$$

denotes the product of two elements of the two-dimensional symplectic group, i.e.,

$$\tilde{\mu} = -\frac{\nu'}{2\nu}\left(1 - \sqrt{1 - 4\mu^2\nu^2}\right) + \mu'\mu, \tag{36}$$

$$\tilde{\nu} = \frac{\nu'}{2\mu}\left(1 + \sqrt{1 - 4\mu^2\nu^2}\right) + \mu'\nu. \tag{37}$$

By means of expression (34) and

$$-\mathrm{i}(\tilde{\mu}q + \tilde{\nu}p) \equiv \alpha\hat{a}^\dagger - \alpha^*\hat{a},$$

we find that

$$\mathcal{K}_\mathcal{S}(n, \mu', \nu', X, \mu, \nu) = \frac{1}{2\pi}\,\mathrm{e}^{\mathrm{i}X}\langle n|\mathcal{D}(\alpha)|n\rangle,$$

where $\mathcal{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$ is the displacement operator. The expansion of the diagonal matrix elements of the displacement operator in the Fock basis is given by [27]

$$\langle n|\mathcal{D}(\alpha)|n\rangle = \mathrm{e}^{-|\alpha|^2/2}L_n(|\alpha|^2),$$

with $|\alpha|^2 = (\tilde{\mu}^2 + \tilde{\nu}^2)/2$. Therefore the distribution function of the discrete random variable n can be related to the symplectic tomogram, which is the probability density distribution of continuous random variable X. This relation reads

$$\mathcal{W}_{\mathrm{sq}}(n, \mu', \nu') = \int \mathrm{d}X\,\mathrm{d}\mu\,\mathrm{d}\nu\,\mathcal{W}_{\mathrm{sym}}(X, \mu, \nu)K_\mathcal{S}(n, \mu', \nu', X, \mu, \nu). \qquad (38)$$

The kernel has the form

$$\mathcal{K}_\mathcal{S}(n, \mu', \nu', X, \mu, \nu) = \frac{\mathrm{e}^{\mathrm{i}X}}{2\pi}\,\mathrm{e}^{-|\alpha|^2/2}L_n(|\alpha|^2). \qquad (39)$$

Here L_n is a Laguerre polynomial and the complex variable α reads

$$\alpha = \tfrac{1}{\sqrt{2}}(\tilde{\nu} - \mathrm{i}\tilde{\mu}), \qquad (40)$$

where $\tilde{\mu}$ and $\tilde{\nu}$ are given by equations (36) and (37).

4.4. Squeeze tomogram and optical tomogram

One can relate the squeeze tomogram $\mathcal{W}_{\mathrm{sq}}(n, \mu, \nu)$ to the optical tomogram $\mathcal{W}_{\mathrm{opt}}(X, \theta)$. To do this one can use the expression obtained above, equation (38), which connects the squeeze tomogram with the symplectic one. Now we relate the symplectic tomogram to the optical one [29]. In fact, one has

$$\mathcal{W}_{\mathrm{opt}}(X, \theta) = \mathcal{W}_{\mathrm{sym}}(X, \cos\theta, \sin\theta). \qquad (41)$$

In view of this, the Fourier decomposition of the optical tomogram can be used to find the symplectic one. In the expresssion

$$\mathcal{W}_{\mathrm{opt}}(X, \theta) = C_0(X) + \sum_{n=1}^{\infty}(C_n(X)\,\mathrm{e}^{\mathrm{i}n\theta} + C_{-n}(X)\,\mathrm{e}^{-\mathrm{i}n\theta}), \qquad (42)$$

where for arbitrary n the coefficients read

$$C_n(X) = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\theta\,\mathcal{W}_{\mathrm{opt}}(X, \theta)\,\mathrm{e}^{-\mathrm{i}n\theta}, \qquad (43)$$

one has the symplectic tomogram by means of the substitutions

$$\cos\theta \to \mu, \qquad \sin\theta \to \nu,$$

or, in more convenient form,

$$\mathrm{e}^{\mathrm{i}n\theta} \to (\mu + \mathrm{i}\nu)^n, \qquad \mathrm{e}^{-\mathrm{i}n\theta} \to (\mu - \mathrm{i}\nu)^n. \qquad (44)$$

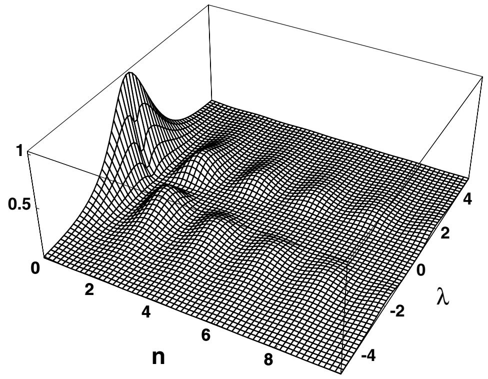

Figure 1. Squeeze tomogram of the vacuum state, which is different from zero only for an even number of photons. Of course, the number of photon is discrete, however, in figures 1–4 it is considered as a continuous label because it makes the figures look nicer.

Thus, one has

$$\mathcal{W}_{\rm sym}(X,\mu,\nu) = \frac{1}{2\pi}\int_0^{2\pi} d\theta' \left[\sum_{n=1}^{\infty}(\mu+i\nu)^n\, e^{-in\theta'}\mathcal{W}_{\rm opt}(X,\theta') + \sum_{n=1}^{\infty}(\mu-i\nu)^n\, e^{in\theta'}\mathcal{W}_{\rm opt}(X,\theta') + \mathcal{W}_{\rm opt}(X,\theta')\right]. \tag{45}$$

Due to scaling transform, there exists another form of the connection of the optical and symplectic tomograms

$$\mathcal{W}_{\rm sym}(X,\mu,\nu) = \frac{1}{\sqrt{\mu^2+\nu^2}}\mathcal{W}_{\rm opt}\left(\frac{X}{\sqrt{\mu^2+\nu^2}}, \arctan\frac{\nu}{\mu}\right).$$

Substituting this expression of the symplectic tomogram into equation (38), we get the explicit relation of the optical tomogram and squeeze tomogram.

5. Examples

In this section we consider several examples of squeeze tomograms of important states of photons. The first example is the ground or vacuum state of the electromagnetic field with density operator

$$\hat{\varrho}_v = |0\rangle\langle 0|. \tag{46}$$

The squeeze tomogram of this state reads

$$\mathcal{W}_0(n,\lambda,\theta) = |\langle n|\hat{S}(\lambda)|0\rangle|^2, \tag{47}$$

where $\hat{S}(\lambda)$ is the unitary squeezing operator. The tomogram in explicit form reads

$$\mathcal{W}_0(n,\lambda,\theta) = \frac{(-\tanh\lambda)^n}{n!2^n\cosh\lambda}\{H_n(0)\}^2. \tag{48}$$

One can see that the angle θ is not present in the tomogram for the vacuum state. In figure 1 we display the squeeze tomogram of the vacuum state, which is symmetric in the parameter λ and goes to zero when the even number of photons increases.

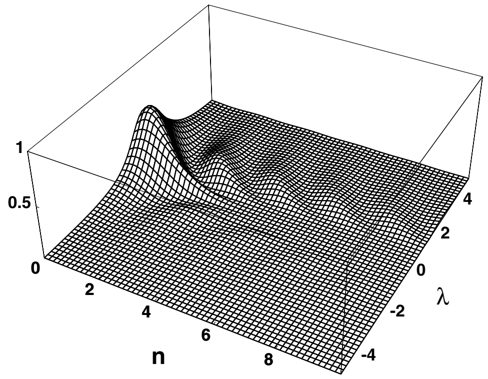

Figure 2. Squeeze tomogram of a coherent state. We used the parameters $\alpha = 1$ and $\theta = 0$, which characterize a non-rotated phase space frame. For $\lambda \to 0$ the tomogram is a photon number Poisson distribution.

Another important state is the coherent state $|\alpha\rangle$ of the photon with the density operator

$$\hat{\varrho}_\alpha = |\alpha\rangle\langle\alpha|. \tag{49}$$

According to definition, the squeeze tomogram for this state reads

$$\mathcal{W}_\alpha(n, \lambda, \theta) = |\langle n|\hat{S}(\lambda)\hat{R}(\theta)|\alpha\rangle|^2. \tag{50}$$

One can easily show that

$$\langle n|\hat{S}(\lambda)\hat{R}(\theta)|\alpha\rangle = \mathrm{e}^{(\lambda+\mathrm{i}\theta)/2}\int \mathrm{d}x\ \psi_n^*(x)\psi_{\tilde{\alpha}}(\mathrm{e}^\lambda x),$$

with $\tilde{\alpha} = \alpha\,\mathrm{e}^{\mathrm{i}\theta}$. Proceeding in the same form as in appendix A we can get

$$\sum_{n=0}^\infty \frac{\beta^{*n}}{\sqrt{n!}}\langle n|\hat{S}(\lambda)\hat{R}(\theta)|\alpha\rangle = \frac{\mathrm{e}^{\mathrm{i}\theta/2}}{\sqrt{\cosh\lambda}}\exp\left\{-\frac{|\alpha|^2}{2} + \frac{1}{2}\tilde{\alpha}^2\tanh\lambda\right\}$$
$$\times\exp\left\{-\frac{1}{2}\beta^{*2}\tanh\lambda + \frac{\tilde{\alpha}}{\cosh\lambda}\beta^*\right\}.$$

By means of the generating function of the Hermite polynomials we obtain the matrix element

$$\langle n|\hat{S}(\lambda)\hat{R}(\theta)|\alpha\rangle = \mathrm{e}^{(-|\alpha|^2+\mathrm{i}\theta+(\alpha\,\mathrm{e}^{\mathrm{i}\theta})^2\tanh\lambda)/2}\sqrt{\frac{|\tanh\lambda|^n}{2^n n!\cosh\lambda}}H_n\left(\frac{\alpha\,\mathrm{e}^{\mathrm{i}\theta}}{\sqrt{|\sinh 2\lambda|}}\right). \tag{51}$$

For $\alpha = 0$, if we take the absolute value of the last expression, we get equation (48). One can see that the tomograms (48) and (50) coincide with the photon distribution function of squeezed vacuum and generic squeezed coherent states, respectively. These photon distributions are given, e.g., in [26, 27]. In figure 2 we illustrate the behaviour of the tomogram of a coherent state as a function of n and λ, using $\alpha = 1$ and $\theta = 0$.

If one considers the tomogram of a squeezed coherent state $|\alpha, r\rangle = \hat{S}(r)|\alpha\rangle$ with the density operator

$$\hat{\varrho}_{\alpha,r} = |\alpha, r\rangle\langle\alpha, r|, \tag{52}$$

the squeeze tomogram of this state reads

$$\mathcal{W}_{\alpha,r}(n, \lambda, \theta) = |\langle n|\hat{S}(\lambda)\hat{R}(\theta)\hat{S}(r)|\alpha\rangle|^2. \tag{53}$$

Using the properties of the product of two elements of the $SU(1,1)$ group we obtain

$$\mathcal{W}_{\alpha,r}(n, \lambda, \theta) = \mathcal{W}_\alpha(n, \lambda + r, \theta). \tag{54}$$

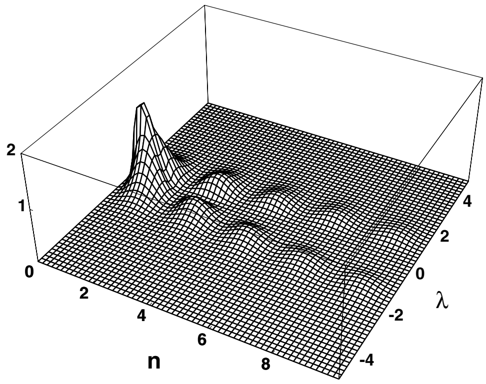

Figure 3. Squeeze tomogram of one photon state. In this case, only for an odd number of photons are the contributions different from zero.

Another specific example is the Fock state of the photon $|m\rangle$ with the density operator

$$\hat{\varrho}_m = |m\rangle\langle m|, \qquad m = 0, 1, \ldots. \tag{55}$$

The squeeze tomogram of this state is

$$\mathcal{W}_m(n, \lambda) = |\langle n|\hat{S}(\lambda)|m\rangle|^2. \tag{56}$$

In fact, it is the modulus squared of the matrix element of the squeezing operator in the Fock basis. It does not depend on the rotation angle θ due to the action of the rotation operator on the Fock state

$$\hat{R}(\theta)|m\rangle = \mathrm{e}^{\mathrm{i}(m+1/2)\theta}|m\rangle. \tag{57}$$

Again, the squeeze tomogram coincides with the photon distribution function of the squeezed Fock state. The example of the tomogram for the Fock state $|1\rangle$ reads

$$\mathcal{W}_1(n, \lambda) = \frac{n^2}{2^{n-1}n!}\frac{(\tanh\lambda)^{n-1}}{(\cosh\lambda)^3}\,[H_{n-1}(0)]^2\,. \tag{58}$$

In figure 3, the tomogram for the one photon state is shown.

The even and odd coherent states (Schrödinger cat states) [22], are paradigmatic examples of the superposition of quantum states. The density operators for these states read

$$\hat{\varrho}_\alpha^{\pm} = |\mathcal{N}_\pm|^2(|\alpha\rangle \pm |-\alpha\rangle)(\langle\alpha| \pm \langle-\alpha|), \tag{59}$$

where

$$\mathcal{N}_\pm = \sqrt{\frac{1}{2(1 \pm \mathrm{e}^{-2|\alpha|^2})}}. \tag{60}$$

The squeeze tomograms for the Schrödinger cat states are

$$\begin{aligned}\mathcal{W}_\alpha^{\pm}(n, \lambda, \theta) &= \frac{1}{1 \pm \mathrm{e}^{-2|\alpha|^2}}\{|\langle n|\hat{S}(\lambda)\hat{R}(\theta)|\alpha\rangle|^2 \pm \mathrm{Re}[\langle n|\hat{S}(\lambda)\hat{R}(\theta)|\alpha\rangle\langle n|\hat{S}(\lambda)\hat{R}(\theta)|-\alpha\rangle^*]\}\\ &= \frac{1}{1 \pm \mathrm{e}^{-2|\alpha|^2}}[1 \pm (-1)^n]\mathcal{W}_\alpha(n, \lambda, \theta).\end{aligned} \tag{61}$$

We observe terms which are due to interference of states $|\alpha\rangle$ and $|-\alpha\rangle$. The tomogram preserves the even and odd character of the states. In figure 4, the even and odd tomograms are displayed for the parameters $\alpha = 0.9$ and $\theta = 0$.

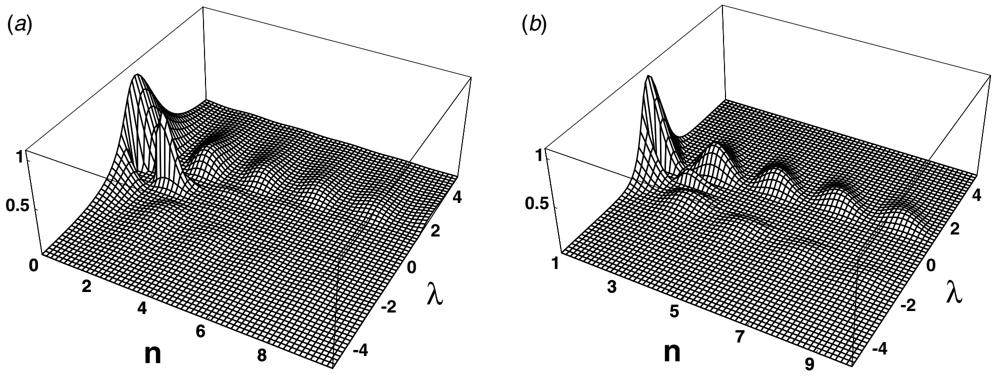

Figure 4. Squeeze tomogram of an even (*a*), and an odd (*b*), coherent state, with parameters $\alpha = 0.9$ and without rotations, i.e., $\theta = 0$. For $\lambda \to 0$, the even tomogram coincides with an even photon number distribution function, while the odd tomogram coincides with an odd photon number distribution function.

The example of a mixed state tomogram (thermal state of light) with the density operator

$$\hat{\varrho}_T = \frac{1}{Z}\mathrm{e}^{-\frac{1}{T}(\hat{a}^\dagger\hat{a}+1/2)},$$
$$Z = \sum_{n=0}^{\infty}\mathrm{e}^{-\frac{1}{T}(n+1/2)} = \frac{1}{2}\mathrm{cosech}\left(\frac{1}{2T}\right), \tag{62}$$

is given by the sum

$$\mathcal{W}_T(n,\lambda,\theta) = \frac{1}{Z}\sum_{m=0}^{\infty}\mathrm{e}^{-\frac{1}{T}(m+1/2)}\mathcal{W}_m(n,\lambda) = \frac{1}{Z}\sum_{m=0}^{\infty}\mathrm{e}^{-\frac{1}{T}(m+1/2)}\frac{\mathrm{sech}(\lambda)}{m!n!}\left[H_{nm}^{\{\mathcal{R}\}}(0)\right]^2. \tag{63}$$

The matrix $\mathcal{R}$ is given by

$$\mathcal{R} = \begin{pmatrix} \tanh\lambda & -\mathrm{sech}\,\lambda \\ -\mathrm{sech}\,\lambda & -\tanh\lambda \end{pmatrix}.$$

One can see that the squeeze tomogram does not depend on the rotation angle θ because it contains the sum of Fock state tomograms, and these tomograms do not depend on the rotation angle. For $T \to 0$, the tomogram is approaching the tomogram of the vacuum state.

6. Extension to multimode case

One can extend the construction of the squeeze tomogram to the case of multimode light. We define the squeeze tomogram in this instance as follows (we focus on two-mode light):

$$\mathcal{W}_{\mathrm{sq}}(n_1,n_2,\mu_1,\mu_2,\nu_1,\nu_2) = \langle n_1 n_2|\hat{\mathcal{S}}(\mu_1,\nu_1)\hat{\mathcal{S}}(\mu_2,\nu_2)\hat{\varrho}\hat{\mathcal{S}}^\dagger(\mu_2,\nu_2)\hat{\mathcal{S}}^\dagger(\mu_1,\nu_1)|n_1 n_2\rangle. \tag{64}$$

Here the operators $\hat{\mathcal{S}}$ are squeeze and rotation operators, which depend on the parameters μ_1, ν_1, μ_2 and ν_2. The integers n_1 and n_2 are the number of photons in each mode correspondingly. The squeeze tomogram (64) is the joint probability distribution function of two discrete random variables. The probability distribution is normalized:

$$\sum_{n_1,n_2=0}^{\infty}\mathcal{W}_{\mathrm{sq}}(n_1,n_2,\mu_1,\mu_2,\nu_1,\nu_2) = 1. \tag{65}$$

One has also the relation

$$\sum_{n_2=0}^{\infty} \mathcal{W}_{\text{sq}}(n_1, n_2, \mu_1, \mu_2, \nu_1, \nu_2) = \mathcal{W}_{\text{sq}}(n_1, \mu_1, \nu_1). \tag{66}$$

The squeeze tomogram for two-mode light is characteristic of the quantum state of light. The squeeze tomogram for the multimode state of photons can be related to the Wigner function, symplectic tomogram and density matrix by the same method as was used in the previous section for one-mode light.

7. Conclusions

To conclude, we summarize the main results of the work.

We introduced a new kind of tomogram which is the photon distribution functions of the field states of light obtained by squeezing and rotation of the density operator. The new tomograms are related to the Wigner function, density matrix and symplectic tomogram by means of the integral transforms. We found the kernels of the integral transforms. In the limit case of identical transform with specific rotation and dilatation parameters, the kernel satisfies the condition to provide the expressions for photon statistics in the chosen initial state in terms of the Wigner function, symplectic tomogram and density matrix.

We extended the approach of squeeze tomography to the multimode case. The obtained results could be complementary characteristics of the quantum states of light in addition to known ones given by optical tomograms and photon number tomograms. The introduced multimode tomograms can be applied to analyse the property of entanglement of the quantum state of light.

The results obtained are mainly connected with theoretical aspects of quantum tomography. We hope that the squeeze-tomography scheme can also be used in the experimental study of light states.

Acknowledgments

MAM and VIM thank the Institute of Nuclear Sciences of UNAM for kind hospitality. MAM thanks the Russian Foundation for Basic Research for partial support under the project no 03-02-16408.

Appendix A. Transformation kernel from density matrix to squeeze tomogram

The squeeze tomogram is defined in equation (18) and if the identity operator in the position representation is introduced, it is given by the expression

$$\mathcal{W}_{\text{sq}} = \int\int \mathrm{d}x\, \mathrm{d}y \langle n|\hat{S}\hat{R}|x\rangle \hat{\varrho}(x, y)\langle y|(\hat{S}\hat{R})^{\dagger}|n\rangle. \tag{A.1}$$

By means of the results

$$\hat{S}|y\rangle = \mathrm{e}^{-\lambda/2}|\mathrm{e}^{-\lambda} y\rangle \tag{A.2}$$

and

$$\hat{S}\hat{R}^{\dagger}\hat{S}^{\dagger} = \exp\left\{-\frac{\mathrm{i}}{2}\theta(\mathrm{e}^{-2\lambda} p^2 + \mathrm{e}^{2\lambda} q^2)\right\}, \tag{A.3}$$

it is immediately clear that

$$\langle y|\hat{R}^{\dagger}\hat{S}^{\dagger}|n\rangle = \mathrm{e}^{-\lambda/2} \int \mathrm{d}y' \psi_n(y') \langle \mathrm{e}^{-\lambda} y| \exp\left\{-\frac{\mathrm{i}}{2}\theta(\mathrm{e}^{-2\lambda} p^2 + \mathrm{e}^{2\lambda} q^2)\right\} |y'\rangle. \tag{A.4}$$

In the last expression, the matrix element within the integration symbol can be identified as a Green function of the operator $H = (\mathrm{e}^{-2\lambda}p^2 + \mathrm{e}^{2\lambda}q^2)/2$, which is given by the expression [27]

$$\mathrm{G}(\mathrm{e}^{-\lambda}y, y', \theta) = \frac{1}{\sqrt{2\pi\,\mathrm{i}\,\mathrm{e}^{-2\lambda}\sin\theta}} \exp\left\{\frac{\mathrm{i}}{2}\cot\theta(y^2 + \mathrm{e}^{2\lambda}y'^2) - \mathrm{i}\,\mathrm{cosec}\,\theta\,\mathrm{e}^{\lambda}yy'\right\}. \tag{A.5}$$

Thus we have that

$$\langle y|\hat{R}^\dagger\hat{S}^\dagger|n\rangle = \mathrm{e}^{\mathrm{i}\cot\theta y^2/2} I_n(y, \mu, \nu), \tag{A.6}$$

with

$$I_n(y, \mu, \nu) = \frac{\mathrm{e}^{-\lambda/2}}{\sqrt{2\pi\,\mathrm{i}\sin\theta}} \int \mathrm{d}y'\,\psi_n(\mathrm{e}^{-\lambda}y') \exp\left\{\frac{\mathrm{i}}{2}\cot\theta y'^2 - \mathrm{i}\,\mathrm{cosec}\,\theta\, yy'\right\}, \tag{A.7}$$

where equation (7),

$$\mu = \mathrm{e}^{\lambda}\cos\theta, \qquad \nu = \mathrm{e}^{-\lambda}\sin\theta,$$

is used to rewrite the expression in terms of the parameters μ and ν. To evaluate the integral in (A.7), one multiplies it by $\beta^n/\sqrt{n!}$ and makes the sum over n; this lets us identify on the right-hand side the coherent state wavefunction in position representation $\psi_\beta(\mathrm{e}^{-\lambda}y')$. This leads to a Gaussian integral which can be evaluated to give

$$\sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}} I_n(y, \mu, \nu) = \frac{\mathrm{e}^{-\beta^2/2}}{\sqrt{\mathrm{i}\sqrt{\pi}(\mu + \mathrm{i}\nu)}} \exp\left\{-\frac{\mathrm{i}}{2}\frac{y^2}{\nu(\mu + \mathrm{i}\nu)}\right\} \times \exp\left\{\mathrm{i}\frac{\nu}{\mu + \mathrm{i}\nu}\beta^2 + \sqrt{2}\frac{y}{\mu + \mathrm{i}\nu}\beta\right\}. \tag{A.8}$$

Next we carry out an expansion in β by means of the generating function of the Hermite polynomials to get

$$\langle y|\hat{R}^\dagger\hat{S}^\dagger|n\rangle = \frac{1}{\sqrt{\mathrm{i}\sqrt{\pi}(\mu + \mathrm{i}\nu)}} \exp\left\{\frac{\mathrm{i}}{2}\left[\cot\theta - \frac{1}{\nu(\mu + \mathrm{i}\nu)}\right] y^2\right\} \times \left(\frac{\mu - \mathrm{i}\nu}{\sqrt{\mu^2 + \nu^2}}\right)^n \frac{1}{\sqrt{2^n n!}} H_n\left(\frac{y}{\sqrt{\mu^2 + \nu^2}}\right). \tag{A.9}$$

Substituting this matrix element and its conjugate with y replaced by x into expression (A.1), we obtain the kernel of the integral transform that connects the squeeze tomogram with the density matrix given in equation (24):

$$\mathcal{K}(x, y, n, \mu, \nu) = \frac{1}{\sqrt{\pi(\mu^2 + \nu^2)}2^n n!} H_n\left(\frac{x}{\sqrt{\mu^2 + \nu^2}}\right) H_n\left(\frac{y}{\sqrt{\mu^2 + \nu^2}}\right) \times \exp\left\{-\mathrm{i}\frac{x^2}{2}\left[\frac{\sqrt{2}}{1 - \sqrt{1 - 4\mu^2\nu^2}} - \frac{\mu + \mathrm{i}\nu}{\nu(\mu^2 + \nu^2)}\right] + \mathrm{i}\frac{y^2}{2}\left[\frac{\sqrt{2}}{1 - \sqrt{1 - 4\mu^2\nu^2}} - \frac{\mu - \mathrm{i}\nu}{\nu(\mu^2 + \nu^2)}\right]\right\}.$$

Appendix B. Tranformation kernel from the Wigner function to squeeze tomogram

The density matrix in position representation can be written in terms of the Wigner function as

$$\varrho(x, y) = \frac{1}{2\pi} \int \mathrm{d}p\, \mathrm{e}^{\mathrm{i}p(x-y)} W\left(\frac{x + y}{2}, p\right). \tag{B.1}$$

Substituting this expression into equation (23) and considering the transformation

$$q = \frac{x+y}{2}, \qquad q' = \frac{x-y}{2},$$

we find equation (25) with

$$\mathcal{K}_W(q, p, n, \mu, \nu) = \frac{1}{\pi}\int \mathrm{d}q'\, \mathrm{e}^{2\mathrm{i}pq'}\mathcal{K}(q+q'q-q', n, \mu, \nu). \tag{B.2}$$

The integral function

$$\mathcal{K}_W(q, p, n, m, \mu, \nu) = \frac{1}{\pi\sqrt{\pi(\mu^2+\nu^2)}}\int \mathrm{d}q'\, \mathrm{e}^{2\mathrm{i}pq'}\frac{1}{\sqrt{2^{n+m}n!m!}}$$
$$\times \exp\{-f_I(q^2+q'^2) - 2\mathrm{i} f_R q q'\} H_n\left(\frac{q+q'}{\sqrt{\mu^2+\nu^2}}\right) H_m\left(\frac{q-q'}{\sqrt{\mu^2+\nu^2}}\right), \tag{B.3}$$

with

$$f_R = \frac{\sqrt{2}}{1-\sqrt{1-4\mu^2\nu^2}} - \frac{\mu}{\nu(\mu^2+\nu^2)}, \qquad f_I = \frac{1}{\mu^2+\nu^2},$$

is defined. It has the property that if $n = m$, one gets (B.2) after the substitution of the transformation kernel given in equation (24).

Multiplying equation (B.3) by $\alpha^n\gamma^m/\sqrt{(n!m!)}$, summing over n and m, and using the expansion of coherent states in terms of the harmonic oscillator wavefunctions,

$$\sum_{n=0}^{\infty}\frac{\alpha^n}{2^{n/2}n!}H_n(x) = \pi^{1/4}\,\mathrm{e}^{|\alpha|^2/2}\mathrm{e}^{x^2/2}\psi_\alpha(x),$$

one has the result

$$I(\alpha,\gamma) \equiv \sum_{n,m=0}^{\infty}\frac{\alpha^n\gamma^n}{\sqrt{n!m!}}\mathcal{K}_W(q, p, n, m, \mu, \nu)$$
$$= \frac{1}{\pi\sqrt{\mu^2+\nu^2}}\,\mathrm{e}^{(|\alpha|^2+|\gamma|^2)/2}\int \mathrm{d}q'\,\mathrm{e}^{2\mathrm{i}pq'}\,\mathrm{e}^{-2\mathrm{i}f_R q q'}$$
$$\times\,\psi_\alpha\left(\frac{q+q'}{\sqrt{\mu^2+\nu^2}}\right)\psi_\gamma\left(\frac{q-q'}{\sqrt{\mu^2+\nu^2}}\right). \tag{B.4}$$

Making the substitutions of the coherent state wavefunctions and evaluating the Gaussian integral, we obtain

$$I(\alpha,\gamma) = \frac{1}{\pi}\exp\left\{-\frac{q^2}{\mu^2+\nu^2} - (\mu^2+\nu^2)(p-f_R q)^2\right\}$$
$$\times\exp\left\{-\frac{1}{2}(\alpha\ \gamma)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\alpha\\ \gamma\end{pmatrix} + (\alpha\ \gamma)\begin{pmatrix}z^*\\ z\end{pmatrix}\right\}, \tag{B.5}$$

with

$$z = \sqrt{\frac{\sqrt{2}}{\mu^2+\nu^2}}q + \mathrm{i}\sqrt{2(\mu^2+\nu^2)}(p-f_R q). \tag{B.6}$$

Using the generating function for the multidimensional Hermite polynomials [27]

$$\exp\left\{-\frac{1}{2}(\alpha\ \gamma)\mathcal{R}\begin{pmatrix}\alpha\\ \gamma\end{pmatrix} + (\alpha\ \gamma)\begin{pmatrix}z^*\\ z\end{pmatrix}\right\} = \sum_{n,m=0}^{\infty}\frac{\alpha^n}{n!}\frac{\gamma^m}{m!}H_{nm}^{\{\mathcal{R}\}}(z^*, z),$$

we have the result

$$\mathcal{K}_W(q, p, n, m, \mu, \nu) = \frac{1}{\pi\sqrt{n!m!}} \mathrm{e}^{-|z|^2/2} H_{nm}^{\{\mathcal{R}\}}(z^*, z),$$
$$\mathcal{R} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{B.7}$$

Finally in [27] appears the result

$$H_{nm}^{\{\mathcal{R}\}}(z^*, z) = (-1)^n n! L_n(|z|^2), \qquad \mathcal{R} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $L_n(x)$ denotes the Laguerre polynomial of order n.

Thus if we consider $n = m$ in expression (B.7) and replace the bidimensional Hermite polynomial, we find the transformation kernel given in equation (26):

$$\mathcal{K}_W(q, p, n, \mu, \nu) = \frac{(-1)^n}{\pi} \exp(-|z|^2/2) L_n(|z|^2),$$

with

$$|z|^2 = \frac{2q^2}{\mu^2 + \nu^2} + 2(\mu^2 + \nu^2)(p - f_R q)^2.$$

References

[1] Schrödinger E 1926 *Ann. Phys., Lpz.* **79** 489
[2] Landau L D 1927 *Z. Phys.* **45** 439
[3] von Neumann J 1932 *Mathematische Grundlagen der Quantenmechanik* (Berlin: Springer)
[4] Wigner E P 1932 *Phys. Rev.* **40** 479
[5] Sudarshan E C G 1963 *Phys. Rev. Lett.* **10** 277
[6] Glauber R 1963 *Phys. Rev. Lett.* **10** 84
[7] Husimi K 1940 *Proc. Phys. Math. Soc. Japan* **23** 264
[8] Bertrand J and Bertrand D 1987 *Found. Phys.* **17** 397
[9] Vogel K and Risken H 1989 *Phys. Rev.* A **40** 2847
[10] Smithey D T, Beck M, Raymer M G and Faridani A 1993 *Phys. Rev. Lett.* **70** 1244
[11] Schiller S, Breitenbach G, Pereira S F, Mikker T and Mlynek J 1996 *Phys. Rev. Lett.* **77** 2933
[12] Mancini S, Man'ko V I and Tombesi P 1995 *Quantum Semiclass. Opt.* **7** 615
[13] D'Ariano G M, Mancini S, Man'ko V I and Tombesi P 1996 *Quantum Semiclass. Opt.* **8** 1017
[14] Wünsche A 2000 *J. Mod. Opt.* **47** 33
[15] Wallentowitz S and Vogel W 1996 *Phys. Rev.* A **53** 4528
[16] Banaszeck K and Wodkiewicz K 1996 *Phys. Rev. Lett.* **76** 4344
[17] Mancini S, Man'ko V I and Tombesi D 1997 *Europhys. Lett.* **37** 79
[18] Dodonov V V and Man'ko V I 1997 *Phys. Lett.* A **229** 335
[19] Man'ko V I and Man'ko O V 1997 *JETP* **85** 430
[20] Klimov A B, Man'ko O V, Man'ko V I, Smirnov Y F and Tolstoy V N 2002 *J. Phys. A: Math. Gen.* **35** 6101
[21] Castaños O, López-Peña R, Man'ko M A and Man'ko V I 2003 *J. Phys. A: Math. Gen.* **36** 4677
Castaños O, López-Peña R, Man'ko M A and Man'ko V I 2003 *J. Opt. B: Quantum Semiclass. Opt.* **5** 227
[22] Dodonov V V, Malkin I A and Man'ko V I 1974 *Physica* **72** 597
[23] Vourdas A and Weiner R M 1987 *Phys. Rev.* A **36** 5866
[24] Schleich W P and Wheeler J A 1987 *Nature* **326** 5866
[25] Gel'fand I M and Shilov G E 1958 *Generalized Functions* vol 1 (Moscow: Nauka) (in Russian)
[26] Schleich W P 2001 *Quantum Optics on Phase Space* (Berlin: Wiley)
[27] Dodonov V V and Man'ko V I 1989 *Invariants and Evolution of Nonstationary Quantum Systems, Proc. P N Lebedev Institute* vol 183 (New York: Nova Science)
[28] Man'ko O V, Man'ko V I and Marmo G 2002 *J. Phys. A: Math. Gen.* **35** 699
[29] Mancini S, Man'ko V I and Tombesi P 1997 *J. Mod. Opt.* **44** 2281